THE LIMIT OF A FUNCTION $z = f(x,y)$ OF 2 VARIABLES

Let $z = f(x,y)$ be a function of two variables and let (a,b) be a particular ordered pair in $\mathbb{R}^2$.

We discuss here what it means to write

$$\text{"} \lim_{(x,y)\to(a,b)} f(x,y) = L \text{"} \text{ and to say}$$

"the Limit of $f(x,y)$, as (x,y) approaches (a,b), exists and is the number L."

Loosely speaking, we are saying here that, as the point (x,y) in the xy plane gets ever closer to (a,b), the function value $f(x,y)$ at the point (x,y) gets ever closer to L.

The official definition is this:

The limit $\lim_{(x,y)\to(a,b)} f(x,y)$ exists and is equal to L

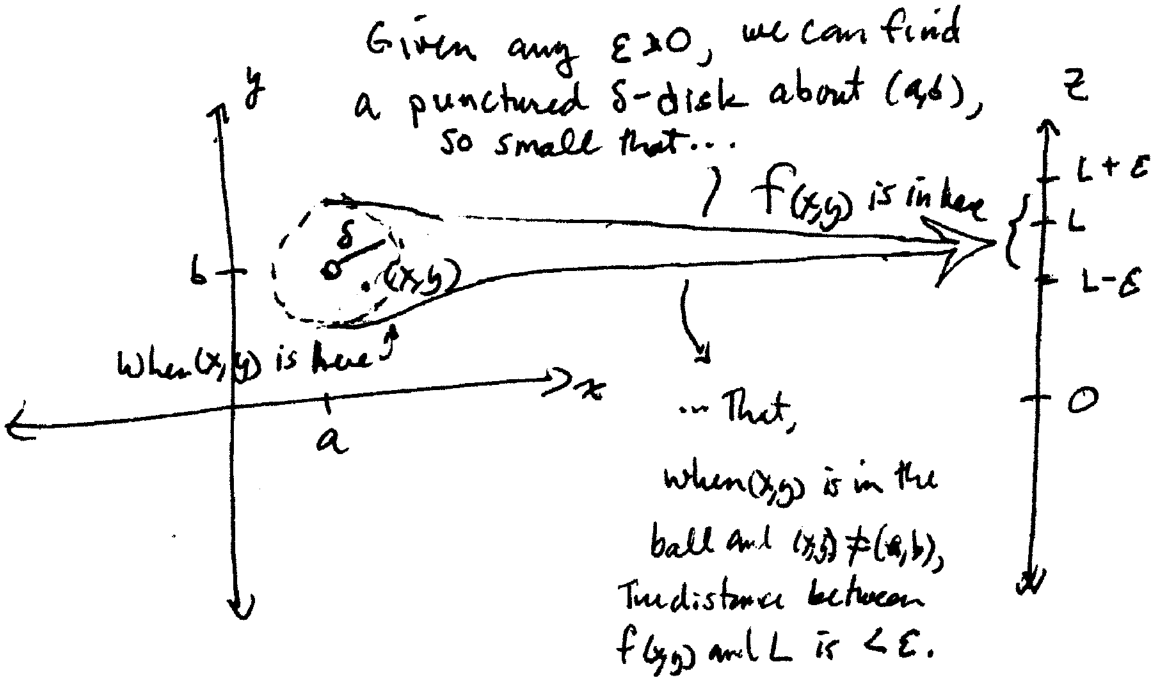
if and only if,

A measure of closeness to L

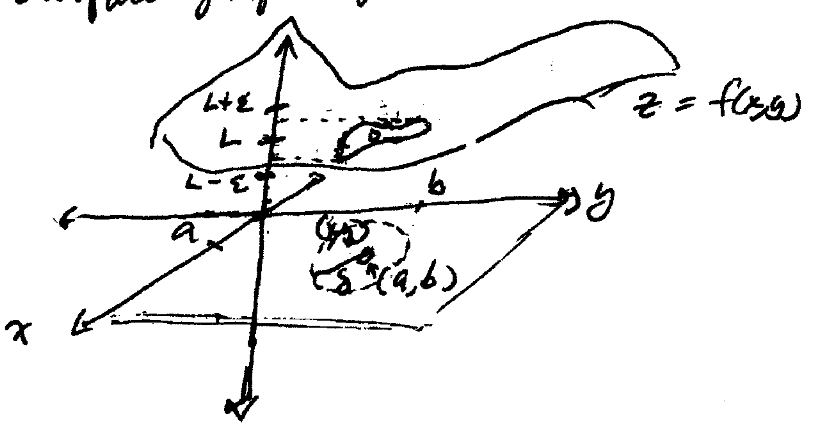
A measure of closeness to (a,b)

for every number $\varepsilon > 0$, there is a $\delta > 0$ such that, whenever (x,y) is a point in the domain D of f such that $0 < \sqrt{(x-a)^2+(y-b)^2} < \delta$, we will have $|f(x,y) - L| < \varepsilon$.

Using an arrow diagram, this definition is illustrated as follows:



This illustration can also be made by consulting the surface graph of the function, $z = f(x,y)$, near the point (a,b)



<u>For example</u>: For $f(x,y) = 1 + x^2 + y^2$,

$1 + 2^2 + 3^2$

$$\lim_{(x,y)\to(2,3)} f(x,y) = \lim_{(x,y)\to(2,3)} (1 + x^2 + y^2) = 14$$

<u>Def'n</u>: If $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$ and $L = f(a,b)$, then

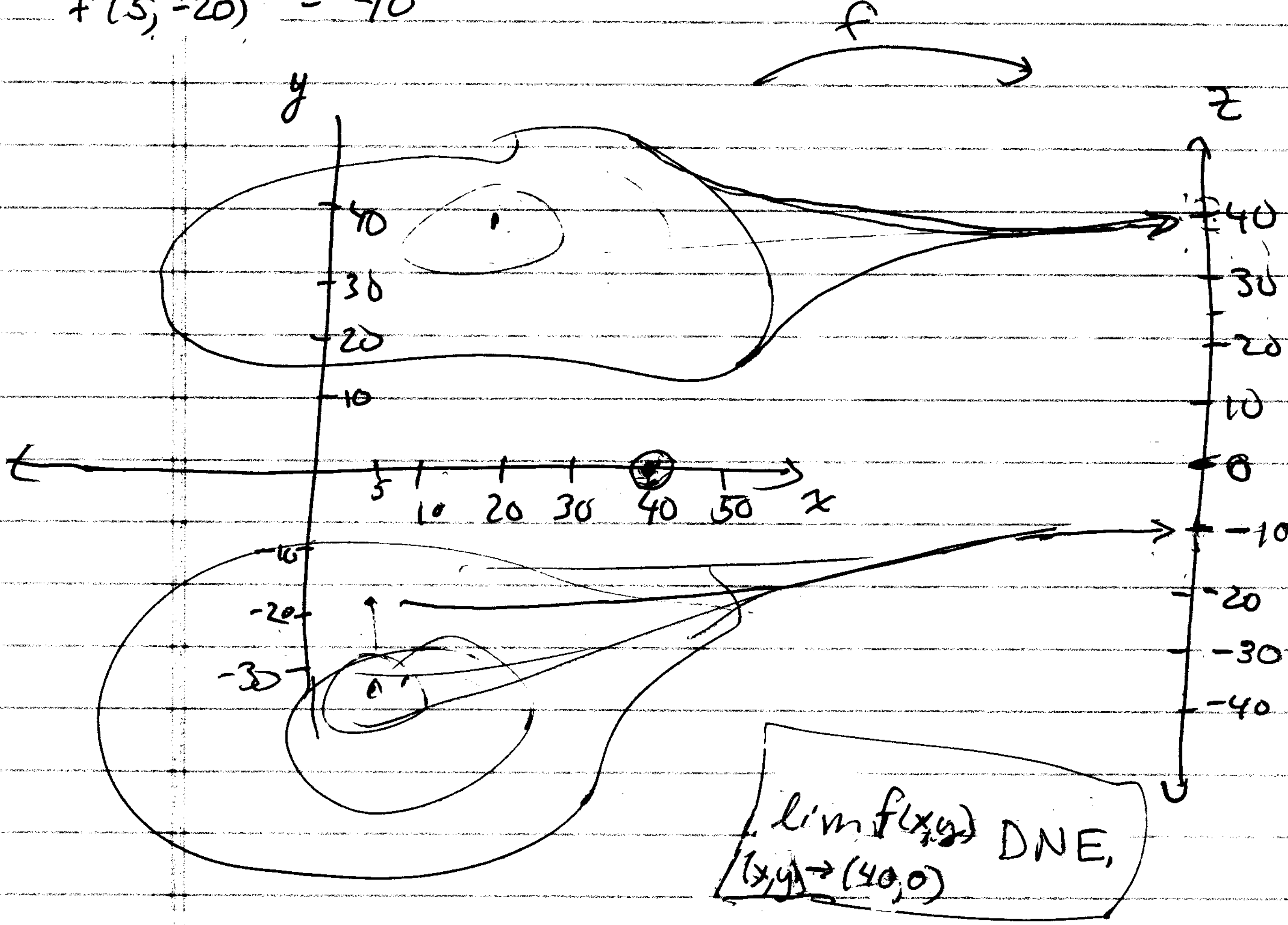
$\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$ and we say f is <u>continuous</u> <u>at $(x,y) = (a,b)$</u>.

AN EXAMPLE WHERE $\lim_{(x,y)\to(a,b)} f(x,y)$ D.N.E.

Define $f(x,y) = \begin{cases} 40 & \text{if } y \geq 0 \\ -10 & \text{if } y < 0 \end{cases}$

$f(25,10) = 40$ since $10 \geq 0$

$f(5,-20) = -10$



$$\lim_{(x,y)\to(10,20)} f(x,y) = 40$$

$$\lim_{(x,y)\to(5,-20)} f(x,y) = -10$$

A Limit Theorem: If $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$ and

$g(t)$ is a function of one variable such that g is continuous at $t = L$, then

$g \circ f(x,y) = g(f(x,y))$ is a function of 2 variables

and $\lim\limits_{(x,y)\to(a,b)} g(f(x,y)) = g(L)$.

For example, Since $\lim\limits_{(x,y)\to(2,3)} (1+x^2+y^2) = 14$,

$$\lim_{(x,y)\to(2,3)} \sqrt{1+x^2+y^2} = \sqrt{14},$$

$$\lim_{(x,y)\to(2,3)} e^{(1+x^2+y^2)} = e^{14}$$

$$\lim_{(x,y)\to(2,3)} \ln(1+x^2+y^2) = \ln 14$$

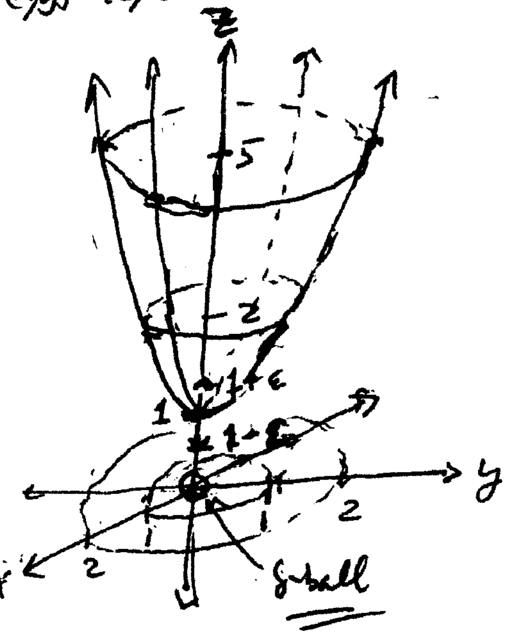
and $\lim\limits_{(x,y)\to(2,3)} \cos(1+x^2+y^2) = \cos 14$.

FACT: When $f(x,y)$ is a <u>polynomial function</u> [like $z = 1+x^2+y^2$] or when $f(x,y)$ is a <u>rational function</u> [like $f(x,y) = \dfrac{x^2+xy+y^2}{x^2+y^2}$], <u>$f$ is continuous at every point (x,y) in its domain</u>.

The following figure illustrates that $\lim_{(x,y)\to(0,0)} 1+x^2+y^2$ exists

and $\lim_{(x,y)\to(0,0)} 1+x^2+y^2 = 1$.

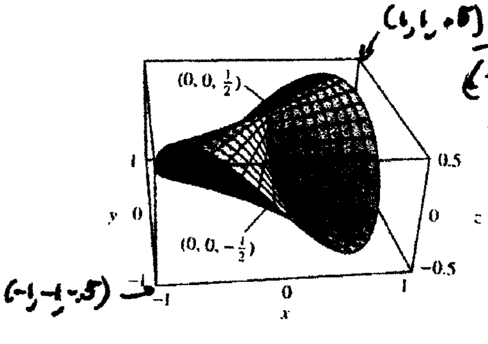
Here, $z = f(x,y) = 1+x^2+y^2$



The function $z = f(x,y) = \dfrac{xy}{x^2+y^2}$ is a function

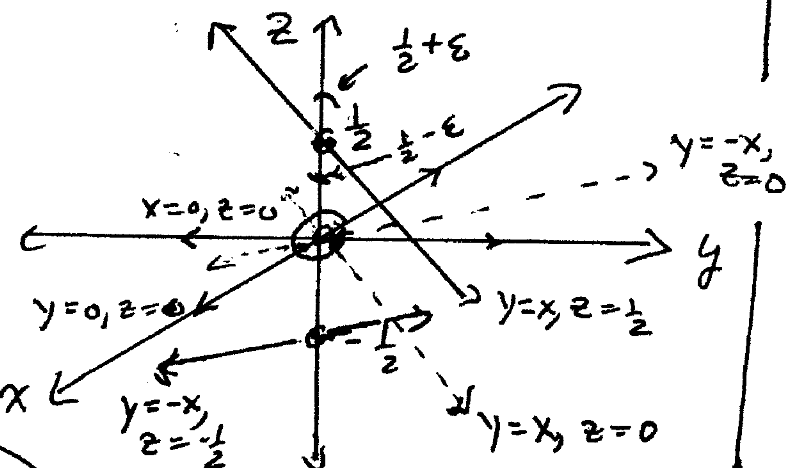
for which $\lim_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ DOES NOT EXIST.

Here is the graph of $y = f(x,y) = \dfrac{xy}{x^2+y^2}$



(The graph is rotated 90° Clockwise in the figure to the right)

ON THIS FIGURE, ONLY THE POINTS ON THE GRAPH such that $y=x$, $y=-x$, $y=0$, or $x=0$ are drawn.



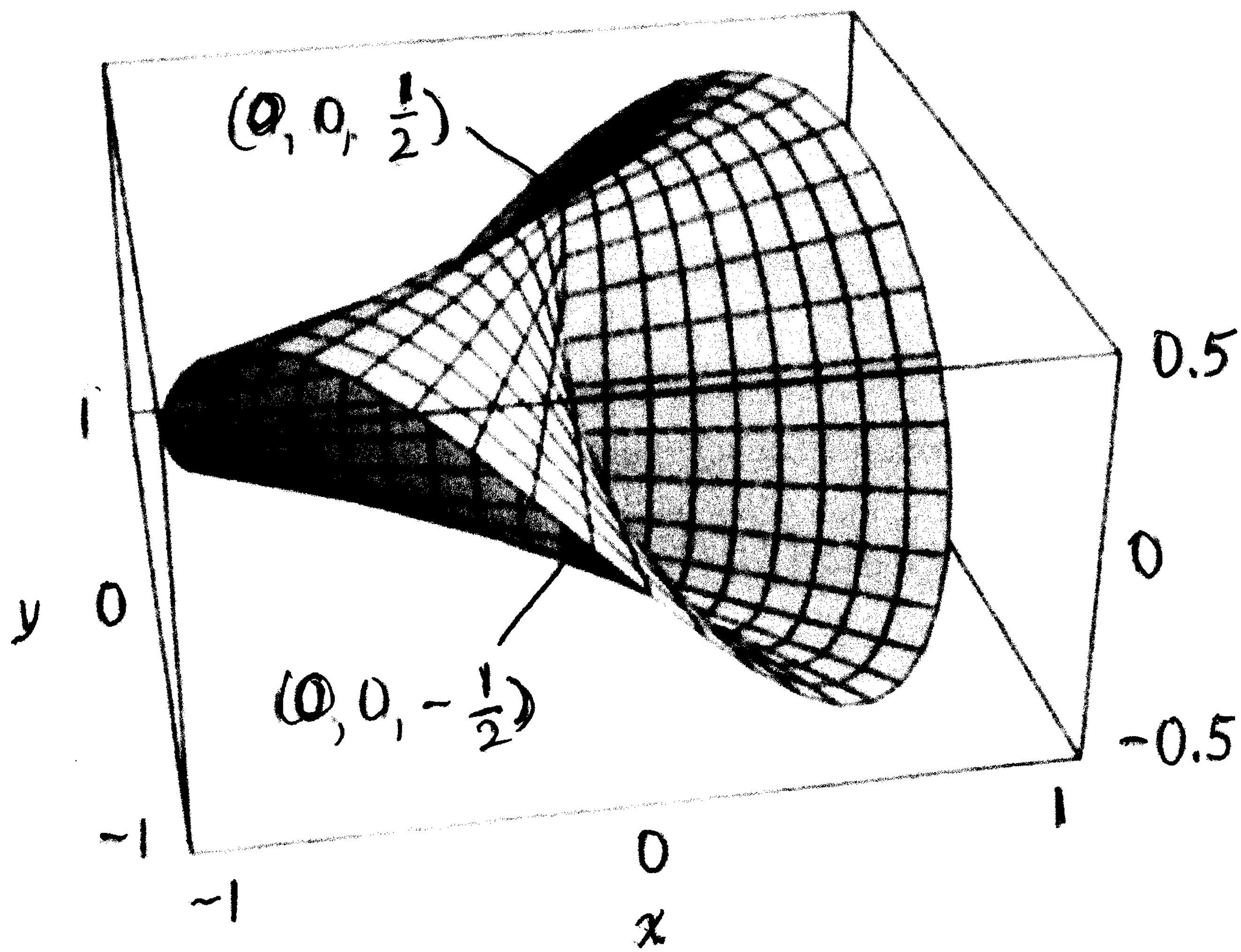
when $y=0$, $f(x,y) = f(x,0) = \frac{0}{x^2} = 0$

when $y=x$, $f(x,y) = f(x,x) = x^2/2x^2 = \frac{1}{2}$

when $y=-x$, $f(x,y) = f(x,-x) = -x^2/2x^2 = -\frac{1}{2}$

when $x=0$, $f(x,y) = f(0,y) = 0/y^2 = 0$

The limit L cannot equal $\frac{1}{2}$ because every δ-ball about $(0,0)$ contains points (x,y) with $z=\frac{1}{2}$, with $z=-\frac{1}{2}$, and with $z=0$.



THE GRAPH OF

$$f(x, y) = \frac{xy}{x^2 + y^2}$$

SEE FIGURE 6 on page 906 of Stewart's "CALCULUS, 8e" for a look at this graph from a different angle.

We can similarly conclude that L cannot equal 0 or $-\frac{1}{2}$, or any other number.

Thus, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ D.N.E. (Does not exist)

Actually, there is an easier way to prove that

$\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ does not exist.

FACT: If $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists and equals L,

then $\lim\limits_{\substack{(x,y)\to(a,b) \\ \text{ALONG ANY} \\ \text{PATH TO } (a,b)}} f(x,y) = L$ and it must be the same limit L for every path by which (x,y) approaches (a,b) and from any direction!

If there are two distinct paths C_1 and C_2 such that $\lim\limits_{\substack{(x,y)\to(a,b) \\ \text{along } C_1}} f(x,y) = L_1$ and $\lim\limits_{\substack{(x,y)\to(a,b) \\ \text{along } C_2}} f(x,y) = L_2$ and $L_1 \neq L_2$,

Then $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ Does NOT EXIST.

This is similar to the Theorem in functions of one variable that says, If $\lim\limits_{x\to a^+} f(x) \neq \lim\limits_{x\to a^-} f(x)$, then $\lim\limits_{x\to a} f(x)$ D.N.E.

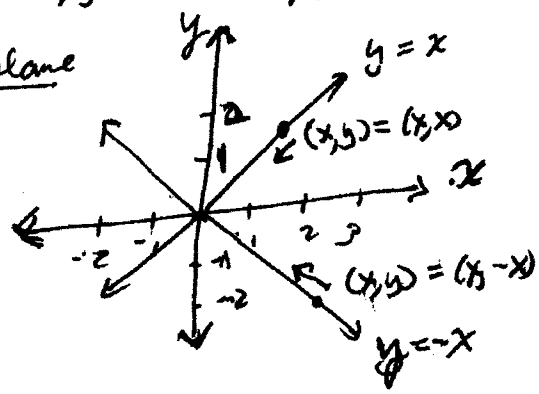
Consider again the function

$$f(x,y) = \frac{xy}{x^2+y^2}.$$

We wish to show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ D.N.E.

In the xy-plane, we look at two paths by which (x,y) can approach $(0,0)$, one along the line $y=x$ and the other along the line $y=-x$.

The xy plane



$$\lim_{\substack{(x,y)\to(0,0)\\ \text{Along } y=x}} \frac{xy}{x^2+y^2} = \lim_{(x,x)\to(0,0)} \frac{x^2}{2x^2} = \frac{1}{2} = L_1$$

$\left(\text{Since } \frac{x^2}{x^2} = 1\right).$

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=-x}} \frac{xy}{x^2+y^2} = \lim_{(x,-x)\to(0,0)} \frac{-x^2}{2x^2} = -\frac{1}{2} = L_2$$

$\left(\text{Since } \frac{-x^2}{x^2} = -1\right)..$

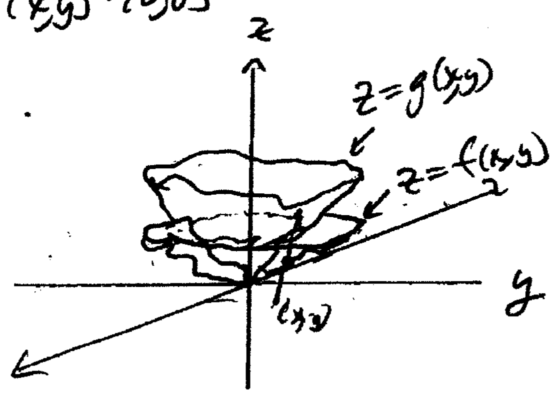
Since $L_1 = \frac{1}{2} \neq L_2 = -\frac{1}{2}$,

$\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ Does Not EXIST.

Sometimes, for a function: $z = f(x,y)$, we can prove that $\lim_{(x,y)\to(0,0)} f(x,y)$ exists if we know that $\lim_{(x,y)\to(0,0)} g(x,y)$ exists for another function $g(x,y)$.

The LITTLE SQUEEZE THEOREM

If $z = f(x,y)$ and $z = g(x,y)$ and for all (x,y) $(x,y) \neq (0,0)$, $0 \leq |f(x,y)| \leq g(x,y)$ and $\lim_{(x,y)\to(0,0)} g(x,y) = 0$, Then $\lim_{(x,y)\to(0,0)} f(x,y)$ exists and $\lim_{(x,y)\to(0,0)} f(x,y) = 0$, too.



The figure to the right should convince you that this is true.

__To Prove:__ $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$ exists and $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$.

__Proof__: Let $f(x,y) = \frac{xy}{\sqrt{x^2+y^2}}$ and $g(x,y) = |x|$.

$$|f(x,y)| = \left|\frac{xy}{\sqrt{x^2+y^2}}\right| = \frac{|y|}{\sqrt{x^2+y^2}}|x| = |f(x,y)|$$

Since $|y| = \sqrt{y^2} \leq \sqrt{x^2+y^2}$, $\frac{|y|}{\sqrt{x^2+y^2}} \leq 1$.; $0 \leq \frac{|y|}{\sqrt{x^2+y^2}} \leq 1$.

So, $0 \leq |f(x,y)| = \frac{|y|}{\sqrt{x^2+y^2}}|x| \leq 1 \cdot |x| = |x| = g(x,y)$.

(proof continued)

Also $\lim_{(x,y)\to(0,0)} g(x,y) = \lim_{(x,y)\to(0,0)} |x| = 0$ since $x \to 0$, as $(x,y)\to(0,0)$.

Then, $0 \le \left| \dfrac{xy}{\sqrt{x^2+y^2}} \right| \le |x|$ $\left[\text{Again, because } \dfrac{|y|}{\sqrt{x^2+y^2}} \le 1\right]$

and $\lim_{(x,y)\to(0,0)} |x| = 0$.

Therefore, by the Little Squeeze Theorem,

$\lim_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}}$ exists and $\lim_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}} = 0$.

Here is another example of a limit calculation:

$$\lim_{(x,y)\to(1,2)} \frac{x^2+x-2}{xy^2-y^2} = \lim_{(x,y)\to(1,2)} \frac{(x-1)(x+2)}{(x-1)y^2} = \lim_{(x,y)\to(1,2)} \frac{x+2}{y^2}$$

$$= \frac{3}{4}, \text{ so } \lim_{(x,y)\to(1,2)} \frac{x^2+x-2}{xy^2-y^2} = \frac{3}{4}.$$
